Differential Geometry I: Curves and surfaces

A. Standard exercises:

7.1 The goal of this exercise is to prove that the differential of the determinant map

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

at $A \in M_n(\mathbb{R})$ is the linear map

$$d(\det)_A : M_n(\mathbb{R}) \to \mathbb{R}$$

given by

$$d(\det)_A(H) = \mathrm{Tr}(\mathrm{Cof}(A)^\top H),$$

where $\mathrm{Cof}(A)$ is the cofactor matrix of A. Recall also from linear algebra that, when A is invertible, we have $A^{-1} = \frac{1}{\det(A)}\mathrm{Cof}(A)^\top$.

We will proceed in three steps:

1. First, prove the formula above or the case $A = I$.
2. Next, assume that $A \in GL_n(\mathbb{R})$, i.e., that A is invertible.
3. Finally, establish the formula for a general matrix $A \in M_n(\mathbb{R})$, using the fact that the matrix $A + tI$ is invertible for t small enough.

   (This is a generally useful trick for identities involving matrices: When required to establish a matrix identity which involves only continuous expressions with respect to the matrix coefficients, it suffices to establish the same identity only for a dense subset of matrices; the subsets of invertible matrices and of diagonalizable matrices are both dense in $\mathcal{M}_n(\mathbb{R})$.)

Solution. (1) Suppose first that $A = I$; we must analyze the expansion of $\det(I + H)$ for a matrix H of sufficiently small norm. Write $n \times n$ matrices as n-tuples of column vectors:

$$I = (E_1, E_2, \dots, E_n), \qquad H = (H_1, H_2, \dots, H_n)$$

where E_j is the j-th vector of the canonical basis. Then, using the properties of the determinant:

$$\begin{aligned}\det(I + H) &= \det(E_1 + H_1, E_2 + H_2, \dots, E_n + H_n)\\ &= \det(E_1, E_2, \dots, E_n) + \det(H_1, E_2, \dots, E_n) + \det(E_1, H_2, \dots, E_n)\\ &\quad \cdots + \det(E_1, E_2, \dots, H_n) + \rho(H)\\ &= \det(I) + \mathrm{Tr}(H) + \rho(H).\end{aligned}$$

Here $\rho(H)$ contains all determinants that have more than two columns H_j, for example $\det(H_1, H_2, \dots, E_n)$. One has $\rho(H) = O(\|H\|^2) = o(\|H\|)$ and consequently $\det(I + H) - \det(I) = \mathrm{Tr}(H) + o(\|H\|)$, and therefore

$$d(\det)_I(H) = \mathrm{Tr}(H).$$

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Another possible reasoning for this first step (starting from the determinant formula):

$$\begin{aligned}\det(I+H) &= \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)(\delta_{\sigma(1)1}+h_{\sigma(1)1})\cdots(\delta_{\sigma(n)n}+h_{\sigma(n)n})\\ &= \delta_{11}\cdots\delta_{nn} + \sum_{i=1}^{n} h_{ii} + \text{terms of degree} \geqslant 2 \text{ in } h_{ij}\\ &= 1+\operatorname{Tr}(H)+o(\|H\|),\end{aligned}$$

hence $d(\det)_I(H)=\operatorname{Tr}(H)$.

(2) Now suppose that $A\in GL_n(\mathbb{R})$. Then by multiplicativity of the determinant and the previous formula we have

$$\begin{aligned}\det(A+H) &= \det(A(I+A^{-1}H))\\ &= \det(A)\det(I+A^{-1}H)\\ &= \det(A)(1+\operatorname{Tr}(A^{-1}H)+o(\|A^{-1}H\|))\\ &= \det(A)+\operatorname{Tr}(\det(A)A^{-1}H)+o(\|H\|)\\ &= \det(A)+\operatorname{Tr}(\operatorname{Cof}(A)^\top H)+o(\|H\|).\end{aligned}$$

Thus $d(\det)_A(H)=\operatorname{Tr}(\operatorname{Cof}(A)^\top H)$.

(3) Finally, if $A\in M_n(\mathbb{R})$ is arbitrary, then for $t>0$ small enough the matrix $A+tI$ is invertible (think about this!). By continuity, we obviously have

$$\operatorname{Cof}(A+tI)\xrightarrow[t\to 0]{}\operatorname{Cof}(A),$$

so $\operatorname{Tr}(\operatorname{Cof}(A+tI)^\top H)\to\operatorname{Tr}(\operatorname{Cof}(A)^\top H)$ and we obtained in complete generality the differential of the determinant:

$$d(\det)_A(H)=\operatorname{Tr}(\operatorname{Cof}(A)^\top H).$$

7.2 Let $\gamma : I\to\mathbb{R}^2$ be a regular plane curve of class C^3 and $r\geqslant 0$. The *parallel curve* to γ at distance r is defined by

$$\gamma_r(t)=\gamma(t)+rN_\gamma(t)$$

(where $N_\gamma=J(T_\gamma)$ is the oriented normal vector field to γ).

(a) Compute the oriented curvature $\kappa_r(t)$ of the parallel curve γ_r (in terms of r and t).

(b) Show that the function $r\mapsto\kappa_r$ satisfies the Riccati differential equation

$$\frac{\partial\kappa}{\partial r}=\kappa^2.$$

(c) Suppose that

$$q = \inf_{t\in I} \frac{1}{|\kappa(t)|} > 0.$$

Show that the map

$$f : (-\varepsilon, \varepsilon) \times I \to \mathbb{R}^2, \quad f(r,t) = \gamma_r(t)$$

is an immersion for all $\varepsilon \leqslant q$.

(d) Make this explicit for the circle of radius a centered at 0.

(e) Explain why the statement in point (c) fails for $\varepsilon > q$.

Remark. This exercise shows in particular that locally, in a neighborhood of the curve, one can construct a curvilinear coordinate system in which one coordinate is the arc length of the curve and the other is the signed distance to the curve. These coordinates are called *Fermi coordinates.*

Solution. (a) Suppose γ is parameterized by arc-length, then (using the Frenet-Serret equations for plane curves):

$$\dot\gamma_r(s) = \dot\gamma(s) + r\dot N(s) = (1 - r\kappa(s))T(s).$$

Hence

$$\ddot\gamma_r(s) = (1 - r\kappa(s))\dot T(s) - r\dot\kappa(s)T(s) = (1 - r\kappa(s))\kappa(s)N(s) - r\dot\kappa(s)T(s).$$

The curvature κ_r of γ_r is therefore given by

$$\kappa_r(s) = \frac{\dot\gamma_r(s) \wedge \ddot\gamma_r(s)}{\|\dot\gamma_r(s)\|^3} = \frac{\kappa(s)}{1 - r\kappa(s)}.$$

(b) It is an elementary calculation:

$$\frac{\partial}{\partial r}\left(\frac{\kappa(s)}{1 - r\kappa(s)}\right) = \left(\frac{\kappa(s)}{1 - r\kappa(s)}\right)^2,$$

(the important point is that this remark illustrates the importance of the Riccati equation in relation with curvature in differential geometry).

(c) The partial derivatives of the map $f(r,s) = \gamma(s) + rN_\gamma(s)$ are

$$\frac{\partial f}{\partial r} = N_\gamma(s), \qquad \frac{\partial f}{\partial s} = (1 - r\kappa(s))T_\gamma(s).$$

These two vectors are linearly independent at every point such that $(1 - r\kappa(s)) \neq 0$; this condition is guaranteed by the hypothesis $|r| < 1/|\kappa(s)|$.

(d) The circle of radius a has curvature $1/a$. The map f is thus defined for $(r,\theta) \in (-a,a) \times [0,2\pi]$ by

$$f(r,\theta) = ((a-r)\cos\theta, (a-r)\sin\theta).$$

Notice that f can be defined on the domain $(-\infty, a) \times [0,2\pi]$, and this map is a simple variant of the polar coordinates of the plane.

(e) The computation in (c) shows that $\frac{\partial f}{\partial s}(r,s)=0$ if $r\kappa(s)=1$. Hence if the domain of f contains such a point, then f is not an immersion. Recall that the point $f(\frac{1}{\kappa(s)},s)=\gamma(s)+\frac{1}{\kappa(s)}N_\gamma(s)$ is the center of the osculating circle of γ at s, and the curve $s\mapsto f(\frac{1}{\kappa(s)},s)$ is the evolute of γ. The evolute of γ, which is also the envelope of its normals, represents a boundary of the domain of regularity of the Fermi coordinates.

7.3 Let $I\subset\mathbb{R}$ be a non-empty open subset. Show that, for any $n\geqslant 2$, there exists no homeomorphism $f:I\to\mathcal{U}$ to an open subset $\mathcal{U}$ of $\mathbb{R}^n$. *(Hint: Check how connecting pairs of points with continuous paths is affected after removing a point from I and from $\mathcal{U}$.)*

Remark. In general, using analogous arguments one can show that a non-empty open set in $\mathbb{R}^n$ is homeomorphic to an open set of $\mathbb{R}^m$ only when $n=m$.

Solution. Assume, for the sake of contradiction, that such a homeomorphism $f:I\to\mathcal{U}$ exists. Note that this implies that $\mathcal{U}\subset\mathbb{R}^n$ has to be path connected: If $q_1,q_2\in\mathcal{U}$ and $p_1=f^{-1}(q_1),p_2=f^{-1}(q_2)\in I$ are their preimages, let $\gamma:[0,1]\to I$ be the segment $\gamma(t)=tp_2+(1-t)p_1$. Then $f\circ\gamma:[0,1]\to\mathcal{U}$ is a continuous curve connecting q_1 to q_2.

Now let $p\in I$ and set $q=f(p)$. Note that, if $\mathcal{V}$ is any open neighborhood of p in I, then $\mathcal{V}\setminus p=\mathcal{V}\cap(I\setminus\{p\})$ is disconnected (since $I\setminus\{p\}$ is the disjoint union of two intervals). Let $\epsilon>0$ be small enough so that $B_\epsilon[q]=\{x\in\mathbb{R}^n:\|x-q\|<\epsilon\}$ is inside $\mathcal{U}$, then $B_\epsilon[q]$ is an open neighborhood of q in $\mathcal{U}$ and, therefore, $\mathcal{V}=f^{-1}(B_\epsilon[q])$ is an open neighborhood of p in I. It is easy to check that f defines a homeomorphism from $\mathcal{V}\setminus\{p\}$ to $B_\epsilon[q]\setminus\{q\}$. However, this is a contradiction, since $B_\epsilon[q]\setminus\{q\}$ is connected, while $\mathcal{V}\setminus\{p\}$ is disconnected.

7.4 **(a)** Is the cone

$$\mathcal{C}=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2=(az)^2\}$$

(with $a\neq 0$) a submanifold of $\mathbb{R}^3$? What about $\mathcal{C}\setminus 0$?

(b) Prove that there exists a differentiable submanifold of $\mathbb{R}^6$ that is homeomorphic to $S^2\times S^2$ (the Cartesian product of two spheres).

(c) Prove that the special linear group

$$SL_n(\mathbb{R})=\{A\in M_n(\mathbb{R})\mid\det(A)=1\}$$

is a differentiable submanifold of $M_n(\mathbb{R})=\mathbb{R}^{n\times n}$. What is its dimension?

(d*) Prove that the orthogonal group $O(n)$ is a differentiable submanifold of $M_n(\mathbb{R})$. What is its dimension?

Remark. The subsets of $GL_n(\mathbb{R})$ that are both subgroups and submanifolds are called the *classical groups.* These are examples of Lie groups (in fact, the most important ones). Examples include $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{R})$ (the symplectic group).

Solution. (a) The cone $\mathcal{C}$ is the set $f^{-1}(0)$ where $f(x,y,z) = x^2+y^2-(az)^2$. The differential of f at (x,y,z) is $df = 2x\,dx + 2y\,dy - 2a^2z\,dz$ (or, equivalently, the gradient is $\nabla f = 2(x,z,a^2)$). This differential is nonzero if $(x,y,z)\neq(0,0,0)$, hence the cone minus the origin $C\setminus\{0\}$ is a submanifold of $\mathbb{R}^3$ (in view of the regular value theorem). However, the whole cone is not a submanifold (in fact, not even a manifold). To see this, consider any neighborhood $U\subset C$ of 0. It is easy to verify that $U\setminus\{(0,0,0)\}$ is not connected, while every point of a submanifold admits connected neighborhoods (since any point point of a sybmanifold has a neighborhood which can be homeomorphically mapped to a neighborhood of 0 in a vector dpace). We conclude that $\mathcal{C}$ is not a (sub)manifold.

(b) Consider the map $f:\mathbb{R}^6\to\mathbb{R}^2$ defined by

$$f(x_1,x_2,x_3,x_4,x_5,x_6) = (f_1(x_1,x_2,x_3),\ f_2(x_4,x_5,x_6)) = (x_1^2+x_2^2+x_3^2-1,\ x_4^2+x_5^2+x_6^2-1).$$

It is easy to verify that $M = f^{-1}(0,0)\subset\mathbb{R}^6 = \mathbb{R}^3\times\mathbb{R}^3$ is homeomorphic to the product of two spheres. We must check that it is a differentiable (i.e. smooth) submanifold. The Jacobian matrix of f is the 2×6 matrix:

$$Df = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x_4 & 2x_5 & 2x_6 \end{pmatrix}.$$

This matrix has rank 2 on the complement of the set

$$S = \{x\in\mathbb{R}^6 \mid x_1=x_2=x_3=0\}\cup\{x\in\mathbb{R}^6 \mid x_4=x_5=x_6=0\}$$

(on S the rank of Df is 1 except at the origin where the rank is 0). In particular the rank of f is constant equal to 2 in a neighborhood of M, hence $M\subset\mathbb{R}^6$ is a differentiable submanifold of codimension 2 (and dimension $6-2=4$).

(c) The special linear group $SL_n(\mathbb{R})$ is the group of $n\times n$ real matrices of determinant 1:

$$SL_n(\mathbb{R}) = \{A\in\mathcal{M}_n(\mathbb{R}) \mid \det(A)=1\}.$$

The determinant map is C^∞ and we have seen in Exercise 7.1 that its differential at A is the linear form

$$H\mapsto d(\det)_A(H) = \mathrm{Tr}(\mathrm{Cof}(A)^T H).$$

For an invertible matrix we always have $\mathrm{Cof}(A)\neq 0$ (since we have $A\cdot\mathrm{Cof}(A)^T = I_n$). In particular, for $H=A$, we have $d(\det)_A(H)\neq 0$. Thus $d(\det)_A$ defines a nonzero linear form on $M_n(\mathbb{R})$ and in particular the determinant map $\det: M_n(\mathbb{R})\to\mathbb{R}$ has constant rank $r=1$ in a neighborhood of $SL_n(\mathbb{R})$. Applying the regular value theorem we deduce that $SL_n(\mathbb{R})\subset M_n(\mathbb{R})$ is a differentiable submanifold of codimension 1 and thus of dimension n^2-1.

(d) The orthogonal group $O(n)$ is the set $O(n)=f^{-1}(0)$ where $f:\mathcal{M}_n(\mathbb{R})\to\mathcal{M}_n(\mathbb{R})$ is the map

$$f(A) = A^TA - I_n.$$

This map is C^∞. We will show that f has constant rank in a neighborhood of $O(n)$. Compute the differential of f at $A\in M_n(\mathbb{R})$. We have

$$f(A+H) = (A+H)^T(A+H) - I_n = A^TA + A^TH + H^TA + H^TH - I_n$$

$$= f(A) + \big(A^T H + H^T A\big) + o(\|H\|).$$

Hence the differential is

$$df_A(H) = A^T H + H^T A.$$

To find the rank of df_A compute its kernel:

$$\ker(df_A) = \{H \in M_n(\mathbb{R}) \mid A^T H = -H^T A\} = \{H \in M_n(\mathbb{R}) \mid A^T H \text{ is antisymmetric}\}.$$

The set of antisymmetric matrices is a vector subspace of dimension $\frac{1}{2}n(n-1)$ of $M_n(\mathbb{R})$. Therefore, if A is invertible (in which case the map $X \to A^T X$ defines a bijective linear map on $\mathcal{M}_n(\mathbb{R})$), then the set of matrices H such that $A^T H$ is antisymmetric is also a vector subspace of dimension $\frac{1}{2}n(n-1)$. Hence

$$\operatorname{rank}(df_A) = n^2 - \dim\ker(df_A) = n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1).$$

Since every matrix of $O(n)$ is invertible, we conclude that the rank of f near $O(n)$ is constant equal to $\frac{1}{2}n(n+1)$. This proves that $O(n) \subset M_n(\mathbb{R})$ is a submanifold of codimension $\frac{1}{2}n(n+1)$ and therefore of dimension

$$\dim(O(n)) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

For example $\dim O(2) = 1$, $\dim O(3) = 3$, $\dim O(4) = 6$.

7.5 In this exercise, we will study submanifolds that arise as level sets of quadratic functions

- **(a)** Recall what a quadratic form Q on a vector space is.
- **(b)** Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Prove that Q is differentiable. What is its differential at a point $x \in \mathbb{R}^n$?
- **(c)** What does Sylvester's theorem from linear algebra state? What is the *signature* of a quadratic form? What does it mean for Q to be *non-degenerate*?
- **(d)** Prove that if $Q : \mathbb{R}^n \to \mathbb{R}$ is a non-degenerate quadratic form, then the hypersurface $Q^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$. What is its dimension?
- **(e)** Is the set
$$S_0(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}$$
a submanifold? The set $S_0(Q)$ is called the *isotropic cone* of the quadratic form Q.
- **(f)** The hypersurfaces
$$S_+(Q) = \{x \in \mathbb{R}^n \mid Q(x) = +1\}, \quad S_-(Q) = \{x \in \mathbb{R}^n \mid Q(x) = -1\}$$
are called the positive and negative *indicatrices* of the quadratic form Q. Show that Q is completely determined by the two indicatrices and the isotropic cone, i.e., if Q_1 and Q_2 are two quadratic forms on $\mathbb{R}^n$ such that
$$S_0(Q_1) = S_0(Q_2), \quad S_+(Q_1) = S_+(Q_2), \quad S_-(Q_1) = S_-(Q_2),$$
then $Q_1 = Q_2$.

Solution. (a) A quadratic form is a map $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x) = B(x, x),$$

where B is a symmetric bilinear form on $\mathbb{R}^n$.

(b) Compute the Taylor expansion of Q at $x \in \mathbb{R}^n$:

$$Q(x + h) = B(x + h, x + h) = Q(x) + 2B(x, h) + Q(h).$$

Clearly $Q(h) = B(h, h) = o(\|h\|)$, hence

$$dQ_x(h) = 2B(x, h).$$

Remark. By definition, the bilinear form B determines the quadratic form Q. The formula just established shows that conversely the symmetric bilinear form B is determined by the quadratic form Q. Recall that B can also be recovered by the polarization formula:

$$B(x, y) = \tfrac{1}{4}\big(Q(x + y) - Q(x - y)\big).$$

(c) Sylvester's theorem guarantees for any symmetric bilinear form B the existence of a basis in which the matrix of B is diagonal with entries $+1$, -1, and 0; i.e. block-diagonal of the form $\mathrm{diag}(I_p, -I_q, 0)$. In that basis the quadratic form writes

$$Q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2.$$

The pair (p, q) is called the signature of Q and $r = p + q$ is the rank. The quadratic form Q is said to be nondegenerate if $r = n$. If $(p, q) = (n, 0)$ we say Q is positive definite, and if $(p, q) = (0, n)$ we say Q is negative definite.

(d) If Q is nondegenerate and $a \neq 0$ then

$$dQ_a = 2B(a, \cdot)$$

is a nonzero linear form. Thus dQ_a has rank $r = 1$. This proves that Q has constant rank 1 on the open set $U = \mathbb{R}^n \setminus \{0\}$, and by the regular value theorem we deduce that $Q^{-1}(c)$ is a hypersurface of $\mathbb{R}^n$ (i.e. a submanifold of codimension 1) for every $c \neq 0$.

(e) The zero vector $0 \in \mathbb{R}^n$ is a critical point of Q. In general the isotropic cone $Q^{-1}(0)$ is not a manifold (except in the trivial case where Q is positive or negative definite and $Q^{-1}(0) = \{0\}$ is a point). For example, if $Q(x, y) = x^2 - y^2$ then the isotropic cone $Q^{-1}(0)$ is the union of two perpendicular lines, which is not a submanifold.

(f) Let Q_1, Q_2 satisfy the assumptions of (f) and let $x \in \mathbb{R}^n$. We must prove $Q_1(x) = Q_2(x)$. Distinguish three cases.

(i) If $\alpha = Q_1(x) > 0$, set $x' = x/\sqrt{\alpha}$. Then $x' \in S_+(Q_1) = S_+(Q_2)$, so $Q_2(x') = 1$ and $Q_2(x) = \alpha Q_2(x') = \alpha = Q_1(x)$.

(ii) If $\alpha = Q_1(x) < 0$, set $x' = x/\sqrt{-\alpha}$. Then $x' \in S_-(Q_1) = S_-(Q_2)$, so $Q_2(x') = -1$ and $Q_2(x) = (-\alpha)Q_2(x') = \alpha = Q_1(x)$.

(iii) If $\alpha = Q_1(x) = 0$, then $x \in S_0(Q_1) = S_0(Q_2)$ and thus $Q_2(x) = 0$.

In all cases $Q_2(x) = Q_1(x)$.

B. Bonus exercise:

7.6 Let $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, where $\{\infty\}$ is an additional point not belonging to $\mathbb{R}^n$. Define a topology on $\widehat{\mathbb{R}}^n$ such that $\mathbb{R}^n$ is open with the usual topology, and the neighborhoods of ∞ are sets of the form $\mathbb{R}^n \setminus K$, where K is compact in $\mathbb{R}^n$.

Now consider the map $f : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ defined by

$$f(x) = \begin{cases} \infty, & \text{if } x = p, \\ p, & \text{if } x = \infty, \\ p + k\dfrac{x-p}{\|x-p\|^2}, & \text{if } x \notin \{p, \infty\}, \end{cases}$$

where p is a point in $\mathbb{R}^n$ and $k > 0$ is a real constant. This map is called the *inversion* of center $p \in \mathbb{R}^n$ and modulus $k > 0$. It plays an important role in geometry and analysis; it has the property that the image of any sphere of $\mathbb{R}^n$ not passing through p is a sphere and the image of any sphere passing through p is a hyperplane.

Answer the following questions:

- **(a)** Describe all convergent sequences in $\widehat{\mathbb{R}}^n$ (no need for a formal proof, just an explanation).
- **(b)** Describe the set of fixed points of f, i.e. $\{x \in \widehat{\mathbb{R}}^n \mid f(x) = x\}$.
- **(c)** Prove that f is a homeomorphism of $\widehat{\mathbb{R}}^n$. What is its inverse? Also prove that f restricts to a diffeomorphism of $\mathbb{R}^n \setminus \{p\}$ onto itself.
- **(d)** Prove that if $n = 2$, f defines an anti-holomorphic map on $\mathbb{C} \setminus \{p\}$.
- **(e)** Compute the differential $df_x(h)$ at a point $x \in \mathbb{R}^n \setminus \{p\}$.
- **(f)** Prove that f is a conformal map on $\mathbb{R}^n \setminus \{p\}$ (a map is called conformal if it preserves angles; concretely, this means proving that df_x is a similarity transformation on $\mathbb{R}^n$). What is the similarity ratio of $df_x(h)$?
- **(g*)** Using (without proof, though you can try also establishing this fact) that the inversion maps spheres not passing through p to spheres not passing through p as well as spheres through p to hyperplanes (with two spheres which are tangent at p being mapped to parallel hyperplanes), can you perform the following construction: Exhibit 9 spheres in $\mathbb{R}^3$, with the property that each of them is tangent to at least 5 of the other spheres, and 3 of those spheres are tangent to at least 7 of the other spheres. Note that two surfaces are tangent to each other if, at their points of intersection, they have the same tangent plane.

Solution. (a) A sequence $\{x_j\}$ in $\widehat{\mathbb{R}}^n$ converges if and only if it is of one of the two types:

(i) either $x_j \neq \infty$ for j large enough and the sequence converges in the usual sense to a point of $\mathbb{R}^n$;

(ii) or it escapes every compact set, i.e. for every $R > 0$ there exists $N \in \mathbb{N}$ such that for all $j \geqslant N$ either $x_j = \infty$ or $\|x_j\| \geqslant R$; in that case the sequence converges to the point ∞.

(b) The fixed points are solutions of $f(x) = x$. This equation can be written

$$x - p = k\frac{x-p}{\|x-p\|^2},$$

i.e.

$$\frac{k}{\|x-p\|^2} = 1,$$

so it is the sphere centered at p of radius $r = 1/\sqrt{k}$.

Remark: Geometrically, the inversion f fixes the sphere centered at p of radius $r = 1/\sqrt{k}$. Moreover it sends interior points of this sphere to its exterior and exterior points into its interior. The inversion thus exchanges interior and exterior of the sphere. It also exchanges the center of the sphere with the point at infinity.

(c) Start by searching for the inverse of f, which will confirm bijectivity. Then we will check continuity.

To find the inverse of f we must solve $f(x) = y$ for x. Clearly if $y = \infty$ then $x = p$, and if $y = p$ then $x = \infty$. Consider the general case $y \notin \{p, \infty\}$. We have

$$y = f(x) \iff y - p = k\frac{x-p}{\|x-p\|^2} = \lambda\,(x-p), \quad \text{with } \lambda = \frac{k}{\|x-p\|^2}.$$

In particular $y - p$ and $x - p$ are proportional. Taking norms in the identity above and using $k > 0$ yields

$$\|y-p\|\,\|x-p\| = k.$$

This relation is symmetric in x and y, hence

$$y = f(x) \iff (y-p) = k\frac{x-p}{\|x-p\|^2} \iff (x-p) = k\frac{y-p}{\|y-p\|^2} \iff x = f(y).$$

Therefore f is bijective and it is its own inverse (an involution).

The function $x \mapsto k/\|x-p\|^2$ is differentiable on $\mathbb{R}^n \setminus \{p\}$, hence so is

$$f(x) = p + k\frac{x-p}{\|x-p\|^2}.$$

Consequently $f = f^{-1}$ is a diffeomorphism of the open set $\mathbb{R}^n \setminus \{p\}$ onto itself.

It remains to show that f is a homeomorphism of $\widehat{\mathbb{R}}^n$: the preimage of a neighborhood of p is a neighborhood of ∞, etc. It is clear from the above that f is continuous at every point of $\widehat{\mathbb{R}}^n \setminus \{p, \infty\}$; we must show that the preimage of a neighborhood of p is a neighborhood of ∞ and vice versa.

Let $W \subset \widehat{\mathbb{R}}^n$ be a neighborhood of p; then there exists $t > 0$ such that

$$W \supset B(p,t) = \{x \in \mathbb{R}^n \mid \|x-p\| < t\}.$$

One checks easily that

$$f^{-1}(W) = f(W) \supset \{y \in \mathbb{R}^n \cup \{\infty\} \mid \|y-p\| > 1/t\} \cup \{\infty\},$$

which is indeed a neighborhood of ∞ by the definition of the topology on $\widehat{\mathbb{R}}^n$. A similar argument shows that the preimage of a neighborhood of ∞ is a neighborhood of p.

One can also reason with convergent sequences: if $\{x_j\} \subset \widehat{\mathbb{R}}^n$ converges to p then $y_j = f(x_j)$ converges to ∞, and if $\{x_j\} \subset \widehat{\mathbb{R}}^n$ converges to ∞ then $y_j = f(x_j)$ converges to p.

We have thus proved that f is bijective, continuous, and its inverse $f^{-1} = f$ is also continuous; therefore f is a homeomorphism of $\widehat{\mathbb{R}}^n$ onto itself.

(d) If we identify $\mathbb{R}^2$ with $\mathbb{C}$ and (x_1, x_2) with $z = x_1 + ix_2$, the inversion writes

$$f(z) = p + \frac{\overline{z-p}}{|z-p|^2} = p + \overline{\frac{1}{z-p}},$$

which is the complex conjugate of the holomorphic map $h(z) = p + 1/(z-p)$. Hence f is anti-holomorphic on $\mathbb{C} \setminus \{p\}$.

(e) There are several ways to compute the differential. One method is to compute

$$df_x(h) = \frac{d}{dt}\bigg|_{t=0} f(x+th).$$

First note

$$\frac{d}{dt}\bigg|_{t=0} \|x+th-p\|^2 = \frac{d}{dt}\bigg|_{t=0} \langle x+th-p, x+th-p\rangle = 2\langle h, x-p\rangle,$$

hence

$$\frac{d}{dt}\bigg|_{t=0} \frac{1}{\|x+th-p\|^2} = -2\frac{\langle h, x-p\rangle}{\|x-p\|^4}.$$

Now compute

$$df_x(h) = \frac{d}{dt}\bigg|_{t=0} k\frac{x+th-p}{\|x+th-p\|^2} = k\left(\frac{h}{\|x-p\|^2} - 2\frac{\langle h, x-p\rangle}{\|x-p\|^4}(x-p)\right).$$

Equivalently,

$$df_x(h) = \frac{k}{\|x-p\|^2}\,h - \frac{2k}{\|x-p\|^4}\langle h, x-p\rangle\,(x-p).$$

(f) Note that if $h \perp (x-p)$ then $df_x(h) = \frac{k}{\|x-p\|^2}h$, and if h is a multiple of $(x-p)$ then $df_x(h) = -\frac{k}{\|x-p\|^2}h$. We conclude that df_x is the composition of the orthogonal reflection across the hyperplane $(x-p)^\perp$ with the homothety of ratio $\lambda = \dfrac{k}{\|x-p\|^2}$. From this, we deduce that df_x is a similarity transformation of ratio $\lambda = \dfrac{k}{\|x-p\|^2}$.

(g*) We will sketch the construction: Let $\mathcal{H}$ be a regular hexagon in the plane $\{z=0\}$ in $\mathbb{R}^3$, and let p_c be its center and $p_1, p_2, \ldots, p_6$ its vertices (labeled in sequential order). Note that if $R = \frac{1}{2}d(p_c, p_1)$, then

$$d(p_c, p_i) = 2R \quad \text{and} d(p_i, p_{i+1}) = 2R \quad \text{for all } i = 1, \ldots, 6\,(\text{with} 6+1 \equiv 1).$$

In particular, if we consider the seven spheres $S_c, S_1 \dots, S_6$ in $\mathbb{R}^3$ with radius R centered, respectively, at $p_c, p_1, \dots, p_6$, then S_c is tangent to S_i for all $i = 1, \dots, 6$ and S_i is tangent to S_c, S_{i-1}, S_{i+1} for $i = 1, \dots, 6$.

Finally, let E_1, E_2 be the two planes at $z = \pm R$, respectively; those are tangent to all seven of the previous spheres.

By applying an inversion with respect to a center that doesn't lie on any of the nine surfaces considered above (hence all spheres and planes considered above will be mapped to spheres, preserving the tangency relations), we obtain the required configuration.